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# Single mass equations for an antisymmetric tensor-bispinor

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**Abstract.** The structure of possible single mass equations for an antisymmetric tensor-bispinor is examined. It is shown that all spin-$\frac{3}{2}$ single mass equations are reducible.

## 1. Introduction

After the discovery of the acausality of the Rarita–Schwinger equation (Velo and Zwanziger 1969) different equations have been adopted for the description of spin-$\frac{3}{2}$ particles. These equations also have different defects. The equations with representations $(\frac{3}{2}, 0) \oplus (1, \frac{1}{2})$ and $(\frac{3}{2}, 0) \oplus (2, \frac{1}{2})$ (Hurley 1971, Loide 1973) are causal in minimal electromagnetic coupling, but have no hermitising matrix. The doubled equation has a hermitising matrix, but describes two spin-$\frac{3}{2}$ particles. As it is shown by Wightman (1976), an instability phenomenon is present in quantisation.

Recently the antisymmetric tensor-bispinor has been used to describe spin-$\frac{3}{2}$ particles (Fisk and Tait 1973, Khalil and Seetharaman 1978, Labonté 1980, 1981). The first equation for an antisymmetric tensor-bispinor was proposed by Fisk and Tait (1973) and it turns out to be causal. As it was shown by Khalil and Seetharaman (1978) the Fisk–Tait equation is reducible and has superfluous representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. Two new equations were proposed by Labonté (1980, 1981). As we shall show later, these equations are also reducible.

In this paper we clarify the possible structure of all single-mass equations for an antisymmetric tensor-bispinor $\psi^{\mu\nu}$. It turns out that all these equations are reducible and therefore the description of spin-$\frac{3}{2}$ particles with the help of the antisymmetric tensor-bispinor is unsatisfactory, because the equations we obtain reduce to simpler equations.

In order to clarify the algebraic structure of possible equations for an antisymmetric tensor-bispinor we use the formalism based on spin projection operators (Loide 1972, Loide and Loide 1977, Biritz 1979). We consider this formalism to be more useful in pure algebraic investigations than the formalism based on Dirac matrices (Fisk and Tait 1973, Labonté 1980, 1981). We also use the results of the papers by Kôiv *et al* (1982a, b) to find the physical parameters and mass spectrum. As we shall see the algebraic structure of the $\beta^{\sigma}$-matrix gives us enough information about the equations and causality.

## 2. Spin-$\frac{3}{2}$ equations for $\psi^{\mu\nu}$

We deal with the first-order wave equations

$$(p_\mu\beta^\mu - m)\Psi = 0, \tag{1}$$

where $\Psi$ is decomposed into the direct sum of irreducible representations of the Lorentz group.

The antisymmetric tensor-bispinor $\psi^{\mu\nu}$ is decomposed as

$$[(1,0)\oplus(0,1)]\otimes[(\tfrac{1}{2},0)\oplus(0,\tfrac{1}{2})]$$
$$= (\tfrac{3}{2},0)\oplus(\tfrac{1}{2},1)\oplus(\tfrac{1}{2},0)\oplus(0,\tfrac{1}{2})\oplus(1,\tfrac{1}{2})\oplus(0,\tfrac{3}{2}).$$

Denoting the representations $1=(\frac{3}{2},0)$, $2=(\frac{1}{2},1)$, $3=(\frac{1}{2},0)$, $4=(0,\frac{1}{2})$, $5=(1,\frac{1}{2})$ and $6=(0,\frac{3}{2})$ we may write $\beta^0$ in the following general form (Loide and Loide 1977)

$$\beta^0 = \begin{vmatrix} 0 & 0 & 0 & 0 & at_{15}^{3/2} & 0 \\ 0 & 0 & 0 & a't_{24}^{1/2} & c(t_{25}^{3/2}+\frac{1}{2}t_{25}^{1/2}) & bt_{26}^{3/2} \\ 0 & 0 & 0 & c't_{34}^{1/2} & b't_{35}^{1/2} & 0 \\ 0 & b't_{42}^{1/2} & c't_{43}^{1/2} & 0 & 0 & 0 \\ bt_{51}^{3/2} & c(t_{52}^{3/2}+\frac{1}{2}t_{52}^{1/2}) & a't_{53}^{1/2} & 0 & 0 & 0 \\ 0 & at_{62}^{3/2} & 0 & 0 & 0 & 0 \end{vmatrix} \tag{2}$$

where $t_{ij}^s$ are spin projection operators and $a$, $b$, $c$, $a'$, $b'$, $c'$ are free parameters which determined the eigenvalues of $\beta^0$.

The hermitising matrix $\Lambda$ is the following

$$\Lambda = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & \rho_1 t_{16}^{3/2} \\ 0 & 0 & 0 & 0 & \rho_2(t_{25}^{3/2}-t_{25}^{1/2}) & 0 \\ 0 & 0 & 0 & \rho_3 t_{34}^{1/2} & 0 & 0 \\ 0 & 0 & \rho_3 t_{43}^{1/2} & 0 & 0 & 0 \\ 0 & \rho_2(t_{52}^{3/2}-t_{52}^{1/2}) & 0' & 0 & 0 & 0 \\ \rho_1 t_{61}^{3/2} & 0 & 0 & 0 & 0 & 0 \end{vmatrix} \tag{3}$$

Derivability from the Lagrangian imposes on the parameters the following conditions

$$a\rho_1 = \rho_2 b^*, \qquad a'\rho_2 = -\rho_3 b'^*, \qquad c, c' \text{ real.} \tag{4}$$

From (4) we see that if we want $\Lambda$ to be non-singular then $a$ and $b$, and also $a'$ and $b'$ must be simultaneously non-zero or equal to zero. So we excluded the barnacled equations, where for example, $a=0$, $b\neq 0$ or vice versa.

It is useful to decompose $\beta^0$

$$\beta^0 = \beta^{3/2} + \beta^{1/2} \tag{5}$$

where $\beta^{3/2}$ is composed with the help of spin projection operators $t_{ij}^{3/2}$ and $\beta^{1/2}$ with the help of $t_{ij}^{1/2}$. As it was shown by Kôiv *et al* (1982a), the investigation of $\beta^{3/2}$ and $\beta^{1/2}$ reduces to the investigation of reduced matrices $\beta_{3/2}$ and $\beta_{1/2}$ formed from free parameters

$$\beta_{3/2} = \begin{vmatrix} 0 & 0 & a & 0 \\ 0 & 0 & c & b \\ b & c & 0 & 0 \\ 0 & a & 0 & 0 \end{vmatrix} \tag{6}$$

$$\beta_{1/2}=\begin{vmatrix} 0 & 0 & a' & d' \\ 0 & 0 & c' & b' \\ b' & c' & 0 & 0 \\ d' & a' & 0 & 0 \end{vmatrix} \tag{7}$$

where $d'=c/2$. In (6) the representations 3 and 4, and in (7) the representations 1 and 6 are omitted, since they do not carry a given spin. The domain of physical parameters and the mass spectrum of matrices (6) and (7) was given by Kôiv *et al* (1982a, b). Here we present the results which give us single mass equations.

In the case of single mass equations, $\beta^0$ satisfies the minimal equation

$$(\beta^0)^L((\beta^0)^2-I)=0 \tag{8}$$

where $L=0, 1, 2, \ldots$. It means that the eigenvalues of $\beta_{3/2}$ and $\beta_{1/2}$ must be $\pm 1$ and 0.

In the case of matrix $\beta_{3/2}$ we have three different choices (Kôiv *et al* 1982a)

$$\text{case I} \qquad ab=1, \quad c=0 \tag{9}$$

$$\text{case II} \qquad a=b=0, \quad c=\pm 1 \tag{10}$$

$$\text{case III} \qquad a=b=c=0. \tag{11}$$

Now we examine the possibilities which determined the spin-$\frac{3}{2}$ part, in more detail.

*Case I.* Equation (1) describes two spin-$\frac{3}{2}$ particles. At the same time the spin-$\frac{3}{2}$ part decomposes into two independent equations for representations $(\frac{3}{2}, 0)\oplus(1, \frac{1}{2})$ and $(0, \frac{3}{2})\oplus(\frac{1}{2}, 1)$.

*Case II.* Equation (1) describes one spin-$\frac{3}{2}$ particle. Since from (10) we have $a=b=0$, the components of representations $(\frac{3}{2}, 0)$ and $(0, \frac{3}{2})$: $\psi_1$ and $\psi_6$ are identically equal to zero. Therefore the representations $(\frac{3}{2}, 0)$ and $(0, \frac{3}{2})$ do not play any role in (1) and we may simply leave them out.

As we have mentioned above we treat the case when $\Lambda$ is nonsingular. If we in (10) take $ab=0$, $a=0$, $b\neq 0$ (or $a\neq 0$, $b=0$) we get the barnacled equations where the representations $(\frac{3}{2}, 0)$ and $(0, \frac{3}{2})$ also do not play an essential role. From (4) we get singular $\Lambda$.

*Case III.* Now we have $\beta_{3/2}=0$ and equation (1) describes only spin-$\frac{1}{2}$ particles. So case III is not of interest to us.

Let us turn to the spin-$\frac{1}{2}$ part which is described by $\beta_{1/2}$. In cases I and II we get different results.

*Case I.* Due to $c=0$ the structure of $\beta_{1/2}$ is exactly the same as for $\beta_{3/2}$. Now the parameters $a'b'$ and $c'$ are independent and we have three different choices

$$\text{Ia} \qquad a'b'=1, \quad c'=0 \tag{12}$$

$$\text{Ib} \qquad a'=b'=0, \quad c'=\pm 1 \tag{13}$$

$$\text{Ic} \qquad a'=b'=c'=0. \tag{14}$$

We discuss them separately.

*Case Ia.* Equation (1) describes two spin-$\frac{3}{2}$ and two spin-$\frac{1}{2}$ particles. At the same time the equation decomposes into two independent equations with linkages

$$(\tfrac{3}{2}, 0)\leftrightarrow(1, \tfrac{1}{2})\leftrightarrow(\tfrac{1}{2}, 0)$$

$$(0, \tfrac{3}{2})\leftrightarrow(\tfrac{1}{2}, 1)\leftrightarrow(0, \tfrac{1}{2}).$$

Since $\beta^0$ satisfies the minimal equation (8), where $L=0$, we have an equation which is equivalent to the Dirac equation for $\psi^{\mu\nu}$

$$(p\gamma - m)\psi^{\mu\nu} = 0. \tag{15}$$

This equation was also written in Labonté (1980). $\beta^0$ is diagonalisable, the equation is causal in minimal electromagnetic coupling.

*Case Ib.* Equation (1) describes two spin-$\frac{3}{2}$ and one spin-$\frac{1}{2}$ particle. At the same time the equation decomposes into three independent equations with linkages

$$(\tfrac{3}{2}, 0) \leftrightarrow (1, \tfrac{1}{2}) \qquad (0, \tfrac{3}{2}) \leftrightarrow (\tfrac{1}{2}, 1)$$
$$(\tfrac{1}{2}, 0) \leftrightarrow (0, \tfrac{1}{2}).$$

Spin-$\frac{3}{2}$ is described by the doubled Hurley equation and the spin-$\frac{1}{2}$ equation is the Dirac equation. $\beta^0$ satisfies (8), where $L=1$. The equation is causal, since $\beta^0$ is diagonalisable (Amar and Dozzio 1975).

*Case Ic.* Equation (1) describes two spin-$\frac{3}{2}$ particles. The bispinor components $\psi_3$ and $\psi_4$ are identically equal to zero. Therefore the representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ are superfluous and we get the doubled Hurley equation with linkages

$$(\tfrac{3}{2}, 0) \leftrightarrow (1, \tfrac{1}{2}) \quad (0, \tfrac{3}{2}) \leftrightarrow (\tfrac{1}{2}, 1)$$
$$(\tfrac{1}{2}, 0) \quad (0, \tfrac{1}{2}).$$

As it was already shown by Khalil and Seetharaman (1978) the equation in case Ic is equivalent to the Fisk–Tait equation. It is the only equation with antisymmetric tensor-bispinor which describes only two spin-$\frac{3}{2}$ particles.

$\beta^0$ satisfies (8), where $L=1$, and the equation is also causal.

In Khalil and Seetharaman (1978) it was claimed that the Fisk–Tait equation is a barnacled wave equation. This is not true, because the representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ are not linked at all and therefore they are not barnacles. The equation we have is simply equivalent to the doubled Hurley equation.

*Case II.* Due to $c \neq 0$ the structure of $\beta_{1/2}$ is different from that of $\beta_{3/2}$. Without the loss of generality we in (10) set $c=1$ and then $d' = \frac{1}{2}$. As we have mentioned above, the representations $(\frac{3}{2}, 0)$ and $(0, \frac{3}{2})$ are superfluous and we have the equations with linkage schema

$$\begin{array}{ccc} (1, \tfrac{1}{2}) & \leftrightarrow & (\tfrac{1}{2}, 1) \\ \updownarrow & & \updownarrow \\ (\tfrac{1}{2}, 0) & \leftrightarrow & (0, \tfrac{1}{2}). \end{array}$$

The equations with representation $(1, \frac{1}{2}) \oplus (0, \frac{1}{2}) \oplus (\frac{1}{2}, 0) \oplus (\frac{1}{2}, 1)$ reduced to the equation for vector–bispinor $\psi^\mu$ and have therefore been previously investigated. We get three different single mass equations (Loide 1977, Kôiv *et al* 1982a, b) which we list as cases IIa, IIb and IIc.

*Case IIa.*

$$a'b' = -\tfrac{1}{4}, \qquad c' = -\tfrac{1}{2}. \tag{16}$$

Equation (1) describes one spin-$\frac{3}{2}$ particle and is equivalent to the well known Rarita–Schwinger (Pauli–Fierz) equation.

Now $\beta_{1/2}$ is nilpotent and $\beta^0$ satisfies (8), where $L=2$. As it is well known (Velo and Zwanziger 1969), the Rarita–Schwinger equation is acausal.

Case IIa gives the only equation with $L = 2$ and is therefore equivalent to the new spin-$\frac{3}{2}$ equation for $\psi^{\mu\nu}$ proposed by Labonté (1980, 1981). As we have shown it is equivalent to the Rarita–Schwinger equation for vector-bispinor $\psi^{\mu}$.

*Case IIb.*

$$a'b' = \tfrac{1}{4}, \qquad c' = \tfrac{1}{2} \quad \text{or} \quad a'b' = -\tfrac{3}{4}, \qquad c' = -\tfrac{3}{2}. \tag{17}$$

The equation (1) describes one spin-$\frac{3}{2}$ and one spin-$\frac{1}{2}$ particle. $\beta^0$ satisfies (8), where $L = 1$. The equation is causal.

*Case IIc.*

$$a'b' = \tfrac{3}{4}, \qquad c' = -\tfrac{1}{2}. \tag{18}$$

The equation describes one spin-$\frac{3}{2}$ and two spin-$\frac{1}{2}$ particles. The given equation is equivalent to the Dirac equation for vector-bispinor

$$(p\gamma - m)\psi^{\mu} = 0.$$

## 3. Conclusions

In this paper we found all the single mass equations for an antisymmetric tensor-bispinor $\psi^{\mu\nu}$. As we have seen they reduced to simpler equations and are not therefore physically interesting. In case I the equation reduces to two or three independent equations. In case II the equation reduces to the equation for vector-bispinor. Therefore if we use the antisymmetric tensor-bispinor, we get nothing new. They give also nothing new in acausality problems.

Using the results of the papers by Kôiv *et al* (1982a, b) it is easy to verify that the antisymmetric tensor-bispinor allows different multi-mass equations. In the multi-mass case the parameters $a$, $b$, $c$, $a'$, $b'$ and $c'$ may be nonzero and we get the irreducible equations.

*Note added in proof.* In this paper we use the formalism based on spin projection operators. The same results may be obtained by the graphical methods given in Cox W 1978 *J. Phys. A: Math. Gen.* **11** 1167–84.

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